Modelling Cluedo: Validity of private announcements

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For our implementation of Cluedo, we needed to implement private announcements [see van Ditmarsch et al., 2007, Chapter 6]. We decided to implement this by removing the relations for the agent that received the announcement between the states where the announcement holds and the states where the announcement does not hold. So if agent i receives announcement φ and R_i is the set of relations before the announcement then R'_i, the set of relations for agent i after the private announcement. R'_i is defined as $R_i = \{(s,t)|(s,t) \in R_i, (M',s) \models \varphi$ if and only if $(M',t) \models \varphi\}$.

We denote propositional variables as p_i^j, where p stands for the category of the card, i for the number of that card in the category and j for the agent that has that card. So p_3^0 means that person card number 3 is dealt to agent 0. In this notation, agent 0 is the envelope. If the number or category of the card is not important, the number is left out and the letter is not p or w. We also have a model M, which after the first public announcement becomes M', after the private announcement becomes M'', and after the second public announcement becomes M'''.

Most of the rules for Cluedo have been implemented in the model implicitly, such as the rule $c^i \to \neg c^j$ with $i \neq j$, which says that only one person can have a specific card. Also the rule that everybody has two cards is implemented in this way.

Assume that we have a set of agents $\mathbf{A}$. Now agent $i \in \mathbf{A}$ speaks an accusation, which is a set of cards denoted as a. Now the agent $j \in \mathbf{A}, i \neq j$ says he has one of these cards, $c \in a$, and shows that card c to i. In our implementation, this is modelled as the public announcement $\bigvee_{k\in a} k^j$. After this public announcement, the private announcement c^j is made towards agent i. After that, the public announcement $\bigvee_{k\in a} E_{i,j} k^j$ is made, to show that both agents know the same card.

During the private announcement, only agent i should get new knowledge, while we would also want that during the public announcements it becomes common knowledge that agent j has at least one of the cards in a and that agent i and agent j now know the same card. This idea is captured in the five propositions.

Proposition 1. $(M''', s0) \models C(\bigvee_{k\in a} \neg k^0)$

Proposition 2. $(M''', s0) \models C_{i,j} \neg c^0$

Proposition 3. $(M''', s0) \models C(\bigvee_{k\in a} C_{i,j} \neg k^0)$

Proposition 4. *For all agents $h \in \mathbf{A}$ with $h \neq i$ and $h \neq j$ it holds that for all* *cards* $k \in a$*:*

$$(M', s0) \models M_h k^0 \textit{ implies } (M'', s0) \models M_h k^0$$

Proposition 5. *For all agents* $h \in \mathbf{A}$ *with* $h \neq i$ *and* $h \neq j$ *it holds that for all cards* $k \in a$*:*

$$(M', s0) \models \neg K_h k^j \textit{ implies } (M'', s0) \models \neg K_h k^j$$

Proof of Proposition 1. First we will prove that $(M', s0) \models \left[\bigvee_{k\in a} k^j\right] C(\bigvee_{k\in a} k^j)$. For this, we first have to see that this announcement is part of the language $\mathcal{L}_N^{u0}$ from van Ditmarsch and Kooi [2006]. This means that we get $\left[\bigvee_{k\in a} k^j\right] \bigvee_{k\in a} k^j$. From propositional logic we now have $\top \to \left[\bigvee_{k\in a} k^j\right] \bigvee_{k\in a} k^j$. Since agent j can only respond to an accusation if it has one of the cards in the accusation, we know that c^j for some $c \in a$. So $\bigvee_{k\in a} k^j$. This means we also get $(\top \wedge \bigvee_{k\in a} k^j) \to E\top$. Now we can apply Proposition 4.26 from van Ditmarsch et al. [2007] to get $\top \to \left[\bigvee_{k\in a} k^j\right] C\left(\bigvee_{k\in a} k^j\right)$. Now by a propositional tautology we get $\left[\bigvee_{k\in a} k^j\right] C\left(\bigvee_{k\in a} k^j\right)$.

Since M' is the model after the first public announcement and we have $(M, s0) \models \left[\bigvee_{k\in a} k^j\right] C\left(\bigvee_{k\in a} k^j\right)$. This means that we also have $(M', s0) \models C\left(\bigvee_{k\in a} k^j\right)$. Since we also have $k^j \to \neg k^0$, via propositional logic we have $\left(\bigvee_{k\in a} k^j\right) \to \left(\bigvee_{k\in a} \neg k^0\right)$, and if we apply (R3) we get $C\left(\left(\bigvee_{k\in a} k^j\right) \to \left(\bigvee_{k\in a} \neg k^0\right)\right)$. Now we can apply modus ponens in order to get $C\left(\bigvee_{k\in a} \neg k^0\right)$. Since the logic PAC is sound and complete with respect to **PAC**, we can conclude that $(M', s0) \models C\left(\bigvee_{k\in a} \neg k^0\right)$. □

Proof of Proposition 2. After we made the private announcement $\langle i, c^j \rangle$, we are in model M''. In this model, there are no relations between states $s, t \in S$ if $(M', s) \models c^j$ and $(M', t) \not\models c^j$ for the agents i and j. This means that there is also no t such that $s0 \twoheadrightarrow_{i,j} t$ with $(M'', t) \models \neg c^j$. Therefore, for all worlds $s0 \twoheadrightarrow_{i,j} u$ we have $(M'', u) \models c^j$. By the definition of C we get $(M'', s0) \models C_{i,j} c^j$. Since we also have $c^j \to \neg c^0$, which holds in each state, so $(M'', s0) \models C_{i,j}(c^j \to \neg c^0)$. Now by modus ponens we get $(M'', s0) \models C_{i,j} \neg c^0$. □

Proof of Proposition 3. Here we make the observation that the second public announcement $\bigvee_{k\in a} E_{i,j} k^j$ is also a member of $\mathcal{L}_N^{u0}$ and that the proof will be similar to the proof of Proposition 1.

After we have gotten $(M''', s0) \models C\left(\bigvee_{k\in a} E_{i,j} k^j\right)$, we now need to promote the E operator to the C operator. For this, we can use the rule $(\varphi \wedge E\varphi) \to C\varphi$ and the rule $E\varphi \to \varphi$. Now by propositional logic we get $\bigwedge_{k\in a} \left(E_{i,j} k^j \to k^j\right)$. Using that and the initial formula we get $(M''', s0) \models C\left(\bigvee_{k\in a} E_{i,j} k^j \wedge k^j\right)$. Now we can conclude $(M''', s0) \models C\left(\bigvee_{k\in a} C_{i,j} k^j\right)$. □

Proof of Proposition 4. We will prove this by contraposition. Take an agent $h \in \mathbf{A}$ such that $h \neq i$ and $h \neq j$ and $(M', s0) \models M_h k^0$ for some $k \in a$. Now assume that $(M'', s0) \not\models M_h k^0$. This means that during the private announcement for i, the relations for h have changed. By the definition of private announcements, this is not what happens, so that means that our assumption did not hold. So $(M', s0) \models M_h k^0$ implies $(M'', s0) \models M_h k^0$ for some $k \in a$. Since k was also chosen arbitrarily, this holds for all cards in a. □

Proof of Proposition 5. We will prove this by contraposition. Take an agent $h \in \mathbf{A}$ such that $h \neq i$ and $h \neq j$ and $(M', s0) \models \neg K_h k^j$ for some $k \in a$. Now assume that $(M'', s0) \not\models \neg K_h k^j$. This means that during the private announcement for i, the relations for h have changed. By the definition of private announcements, this is not what happens, so that means that our assumption did not hold. So $(M', s0) \models \neg K_h k^j$ implies $(M'', s0) \models \neg K_h k^j$ for some $k \in a$. Since k was also chosen arbitrarily, this holds for all cards in a. $\square$

References

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